*1.5 Reciprocity and Its Application*

A mere four years after the publication of Tellegen’s theorem, J. L. Bordewijk (a student of Professor Tellegen’s at the Technical University of Delft) used his theorem to define and extend the concept of reciprocity. An *N*-port circuit is called *reciprocal* if its port voltages and currents satisfy the relation *∑ ik.vk*’ *= ∑ ik*’*vk,.* Here, the variables *ik* and *vk* are the port currents and voltages and currents under one set of excitations, while *ik*’ and *vk*’ exist under a different one, f*or the same internal branches of the N-port.* The summations are for all *N* ports. By Tellegen’s theorem, the same relation must then hold for the internal branches of the *N-*port [1.3].

The condition and its applications will next be analyzed for two-ports.

Consider the circuits shown in Fig. 1.17. Assume that the two-port contains only linear resistors, and no controlled sources. Regarding the circuit of Fig. 1.17(a) as ***N*** and that of Fig. 1.17(b) as ***N’***, Tellegen’s theorem gives



The two transfer functions (*transconductances)* are thus equal. Note that in this analysis **N** and **N’** contain the same internal branches, but under different terminations. Note also that assuming reciprocity for the two-port, the relation $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k}$gives directly $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$,



 Figure 1.17: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

Consider next the circuits shown in Fig. 1.18. Assuming that the two-port is reciprocal, combining $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k} $with $i\_{2}=i\_{1}^{'}=0$ gives $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$ and ${v\_{2}}/{i\_{1}}={v\_{1}^{'}}/{i\_{2}^{'}}$. However, to use these relations for finding the relation between the transfer functions, the input impedance of **N** andthe output impedance of **N’** must also be found. Note in particular that ${v\_{2}}/{v\_{1}} $is *not* equal to ${v\_{2}^{'}}/{v\_{1}^{'}}$. In general, the forward transfer of **N e**quals the reverse transfer function of **N’**only if the source impedances at the ports are the same. This is valid for the circuits shownin Fig. 5. Analysis gives ${v\_{2}}/{v\_{1}}={i\_{1}^{'}}/{v\_{1}^{'}}$.



 Figure 1.18: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

The forward transfer of **N e**quals the reverse transfer function of **N’**only if the source impedances at the ports are the same. This is valid for the circuits shown in Figs. 1.17 and 51.19, but not for Fig. 1.18. 

Figure 1.19: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

The circuits which meet the equal-impedancetermination condition are shown in Fig. 1.20.

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Figure 1.20. Reciprocity relations for passive linear circuits. Redraw figure! With no writing

For the steady-state sine wave analysis of a linear circuit, all voltages and currents can be represented by their complex phasors, *V(jω)* and *I(jω),* and the passive elements by their impedances *Zk(jω).* Since the KCL and KVL holds for the phasors, all derivations given above for the time-domain quantities can be replicated, to give the same relations between the voltage and current phasors of reciprocal networks as the ones derived above in the time domain.

A useful application of reciprocity is in the analysis of passive networks with multiple excitations. Such circuits occur, e.g. in the design of digital-to-analog data converters (DACs). Consider the circuit of Fig. 1.21(a). It contains five independent sources. Hence, a straightforward approach to its analysis would be to use superposition. This would require five analyses, each with one of the sources active and all others set to zero. Alternatively, node analysis may be used. This would require solving simultaneous equations involving all current sources.



Figure 1.21: (a) Physical network **N;** (b) Auxiliary network **N’.**

**Redo without green**

By contrast, using reciprocity the contribution of *V*1 to the output *vout* can be found from Fig. 1.21(b) using the relation

$$\begin{array}{c}\frac{v\_{out}}{V\_{1}}=\frac{i\_{1}}{I} \#\left(1.27\right)\end{array}$$

and those of the current sources *I*k from the relations

$$\begin{array}{c}\frac{v\_{out}}{I\_{k}}=\frac{v\_{k}}{I} k=1, 2, 3, 4, 5\#\left(1.28\right)\end{array}$$

Note that this calculation requires only *one* circuit analysis, that of the circuit of Fig. 1.21(b).

Generalizing the method, the analysis requires the following steps:

1. Redraw the circuit, setting the values of all independent sources to zero. Thus, voltage sources become short circuits, current sources become open circuits.
2. Replace the output signal with an independent source. If it is voltage, replace it with a current source *I* ; if it is a current, with a voltage source *V.* For convenience, *I* = -1 amp, or *V*= 1 volt may be used.
3. Analyze the transformed network. The desired result will be the weighted sum of the independent source values in the physical circuit, with the weight factor of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica. These weight factors also give the gains of the circuit associated with each source.

*1.6. Interreciprocity and Its Application*

If the circuit contains a controlled source, the reciprocity conditions fail. Consider a linear circuit containing resistors and also the voltage-controlled current source (VCCS) shown in Fig. 1.22. Now the condition for the internal branches

$$\begin{array}{c}\sum\_{}^{}v\_{k}i\_{k}^{'}=\sum\_{}^{}v\_{k}^{'}i\_{k} \#\left(1.29\right)\end{array}$$

will no longer hold for general excitations. However, a modified form of reciprocity, called *interreciprocity* may be introduced for such non-reciprocal circuits. It also offers useful applications, similar to the ones available for reciprocal two-ports. It involves the introduction of a modified version **N’** of the physical network **N,** called the *adjoint network.* It is constructed such that the reciprocity condition on the port currents and voltages

$$\begin{array}{c}i\_{1}v\_{1}^{'}+i\_{2}v\_{2}^{'}=i\_{1}^{'}v\_{1}+i\_{2}^{'}v\_{2} \#\left(1.30\right)\end{array}$$

is restored for the two networks. By Tellegen’s theorem, this requires that the condition

$$\begin{array}{c}\sum\_{}^{}v\_{k}i\_{k}^{'}=\sum\_{}^{}v\_{k}^{'}i\_{k} \#\left(1.31\right)\end{array}$$

also remain valid for the internal branches of **N** and **N’** under all excitations**.** Thus, **N’** must be changed so as to achieve this.

Let the two-port contain resistors and a single voltage-controlled current source located in branches *a* and *b* (Fig. 1.22(a)). Its adjoint replica, unknown at this point, is shown in Fig. 1.22(b). In order to satisfy eq. (1.31), the following relation between the voltages and currents in branches *a* and *b* in **N** and **N’** must hold

$$\begin{array}{c}i\_{a}v\_{a}^{'}+i\_{b}v\_{b}^{'}=i\_{a}^{'}v\_{a}+i\_{b}^{'}v\_{b} \#\left(1.32\right)\end{array}$$

This can be used to find the two branches in **N’** corresponding to the VCVS**.** For the VCCS, the branch relations are $i\_{a}=0$, and $i\_{b}=g\_{m}v\_{a}$*.* Substituting into (1.32) gives the reciprocity condition

 $\begin{array}{c}i\_{a}^{'}-g\_{m}v\_{b}^{'}=-i\_{b}^{'}\frac{v\_{b}}{v\_{a}} \#\left(1.33\right)\end{array}$



1. (b)

Fig. 1.22. VCCS and its adjoint circuit.

Eq. (1.33) must hold for *all* values of ${v\_{b}}/{v\_{a}}$. This is only possible if $i\_{b}^{'}=0$, and $i\_{a}^{'}=g\_{m}v\_{b}^{'}$ *ia*. The image in **N’** of the VCCS of circuit **N** is thus another VCCS, turned around. If there are several VCCS’s in **N,** then the adjoint network **N’** must contain such appropriately modified replicas (i.e., turned around VCCS stages) for all of them.

Generalizing this result, consider the branch admittance matrix ***G***connecting the internal branch voltages and XX Please fix this paragraph.

currents in **N** according to the relation $i\_{B}=Gv\_{B}$. For a two-port containing only resistors and VCCS stages,the resistive branches will be represented in ***G*** by the diagonal elements 1/*R*1 , 1/*R2* , …. A VCCS *gmi* in which branch *m* controls the current in branch *l* introduces a 0 at the location (*m,l*) and the entry *gmi* in (*l,m*). Its replica VCCS in **N’** introduces a 0 at the location (*l,m*) and the entry *gmi* in (*m,l*). This occurs for all VCCS blocks in **N.** Clearly, the branch admittance matrix ***G’*** of ***N’*** is the transpose of ***G.***

The derivation given above can be simplified. For a two-port containing only resistors and VCCSs, eq. (1.31) may be written in the form

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{'T}Gv\_{B} \#\left(1.34\right)\end{array}$$

Since the quantities on both sides are scalars, it is permissible to take the transpose of the right side. This gives

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{T}G^{T}v\_{B}^{'}\#\left(1.35\right)\end{array}$$

and hence $G^{'}=G^{T}$, as obtained before.

A derivation similar to the one resulting in the adjoint branches of the VCCS can be performed to give the adjoint branches of all controlled sources. A table of the results is shown in Fig. 1.23. (Sh. 21 or T-LaPatra Table 9.1).



Fig. 1.23: Controlled sources with their adjoint networks.

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By the definition of **N’,** the physical network **N**and its adjoint network **N’** satisfy the reciprocity condition. Hence, the efficient analysis derived in Sec. 1.2 for multi-source active networks can be used with a slight modification, as described below:

1. Draw the adjoint circuit **N’**, setting the values of all previous independent sources to zero. Thus, voltage sources become short circuits, current sources open circuits.
2. Replace the original output signal with an independent source. If it is voltage, replace it with a current source *I*; if it is a voltage source, with a voltage source *V.* For convenience, *I* = -1 amp, of *V*= 1 volt may be used.
3. Analyze the *adjoint network* **N’**. The desired result for the output in **N** will be the weighted sum of the independent source values in **N** with the weight of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica.

Fig. 1.24 illustrates the process for a network **N** with many independent sources and an output voltage.





Figure 1.24. The application of interreciprocity for the analysis and Thevenin modeling of multi-source circuits.

Interreciprocity may also be used to obtain efficiently the Thevenin equivalent of a linear circuit with several independent as well as dependent sources. The open-circuit output *vo* is the voltage in the Thevenin model, derived as shown above.From Fig. 1.24, the output impedance *Z* of the physical network N can be obtained by setting all independent sources to 0, and finding the output voltage when the output port is excited by a -1 amp current source. By interreciprocity, $v\_{o}^{'}i\_{o}=i\_{o}^{'}v\_{o}$. Hence, the impedance in the Thevenin model is numerically equal to the output voltage of **N’,** thus $Z=Z^{'}=-v\_{o}^{'}$.

There are several more applications of interreciprocity and the adjoint network. One is noise analysis in linear or nonlinear active circuits, another is sensitivity analysis. These will be briefly discussed next.

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*1.7. Noise Analysis Using Interreciprocity*

In a CMOS integrated circuit, every transistor is affected by thermal noise caused by the random motion of the charge carriers in its channel, and by flicker noise (1/f noise) due to the trapping and releasing of these carriers. These noise effects can be modeled by a single independent noise voltage source at the gate of the device (Fig. 11a). Since there may be dozens or even hundreds of devices in circuit, the direct calculation of the overall output noise would be extremely time consuming. Interreciprocity offers an economical alternative [1.4]. For noise calculation, each transistor can be modeled with a transconductance *gm* and the drain-to-source conductance *gds* (Fig. 11b)*.* In the adjoint network, the transconductance is turned around (Fig. 11c). Using interreciprocity, the contribution of the noise voltage *vn* to the output noise power will be $j\_{n}^{'2}=\left[g\_{m}v\_{ds}^{'}\right]^{2}v\_{n}^{2}$. Here, *vn2* is the power (mean square value) of the noise. Hence, a single analysis of the adjoint network to determine the voltages across the *gds* resistors enables the designer to find the total output noise power.



REDRAW (capital N on top)

 Fig. 1.25. (a) Noisy transistor; (b) Its small-signal model; (c) Its adjoint model.

*1.8. Sensitivity Analysis Using Interreciprocity*

In the actual implementation of a circuit, the values of its components will deviate from their theoretical ones. It is important to know how the behavior of the circuit is affected by these errors. Thus, it is useful to be able to calculate the sensitivities of the key performance parameters to element value variations. The use of the adjoint network to do this efficiently was suggested by Director and Rohrer [1.5],[1.6]. Consider again a two-port containing resistors and VCCSs, and assume that all component values have a small error, causing **G** to change to **G + ∆G.** Using the adjoint network **N’** with its transconductance matrix **GT**, eq. (1.35) is replaced by

$$\begin{array}{c}v\_{B}^{T}\left[G+∆G\right]^{T}v\_{B}^{'}-v\_{B}^{T}G^{T}v\_{B}^{'}=v\_{B}^{T}∆G^{T}v\_{B}^{'}\#\left(1.36\right)\end{array}$$

Eq. (19) gives the expression $\sum\_{}^{}i\_{k}v\_{k}^{'}-\sum\_{}^{}i\_{k}^{'}v\_{k}$ for the internal branches of two-port **N.***,* By Tellegen’s theorem, this is the negative of the same expression for the port variables. In the latter, $i\_{k}^{'}$and $v\_{k}^{'}$ remain unchanged, but*ik*and *vk*change, due to the variations in the element value.Equating the two expressions, the changes in the port variables are easily found. These give the sensitivities of the output voltage or current to all circuit parameters.

The process is illustrated with an example [1.7, pp. 389-390].

Assuming a voltage output *vout* for the physical circuit **N** containing resistors and VCCSs, the process is as follows:

1. Construct the adjoint network N’, in which the values of all independent sources of N are set to zero. Resistive branches remain unchanged, controlled sources replaced by their adjoint models. At the output port, place a current source *I =* 1 V.
2. Calculate the branch voltages $v\_{k}^{'}$across all resistors. According to the discussions above, the contribution of the incremental change *∆Gk* of an admittance *Gk* to *vout* will be $v\_{k}^{'}v\_{k}∆G\_{k}$. Thus, the output sensitivity to variations in *Gk* will be ${∆v\_{out}}/{∆G\_{k}}=v\_{k}^{'}v\_{k}$.
3. To find the sensitivity to changes in the transconductance *Glm* of a VCCS between branches *m* and *l* (Fig. X) (Fig. 9.9 in T-laP), find the controlling voltages $v\_{m}^{ }$ and $v\_{l}^{'}$in N and N’. The sensitivity is ${∆v\_{out}}/{∆G\_{lm}}=v\_{l}^{'}v\_{m}$.

Note that the formula given above has a physical interpretation. The current *ik* is the result of signal transfer from the independentsources in N to branch *k,* while $i\_{k}^{'}$ the transfer in N’ from the 1 amp current source at the former output port N’. In N, this corresponds to signal transfer from branch *k* to the output. The sensitivity is the product of these two transfer functions.

Note also that all sensitivities are obtained simultaneously in one circuit analysis, that of N’.

Sensitivity calculations for other types of controlled sources can be performed in similar ways. Consider the case of a voltage-controlled current source in **N** (Fig. 20a). In the adjoint network **N’**, it is translated into a similar VCCS, but with its input and output terminals interchanged (Fig.20b) A small change in the transconductance *Gm* can be modelled by an added current source $v\_{k}∆G\_{m}$ in parallel with the main one. Using interreciprocity, the effect of this added source will be an added output term of value $v\_{l}^{'}v\_{k}∆G\_{m}$. Here, $v\_{l}^{'}$ is the controlling voltage of the VCCS in **N’**. Hence, the sensitivity to errors in *Gm* is the product of the controlling voltages in **N** and **N’.**

*References to Chapter 1*

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*1.5 Reciprocity and Its Application*

A mere four years after the publication of Tellegen’s theorem, J. L. Bordewijk (a student of Professor Tellegen’s at the Technical University of Delft) used his theorem to define and extend the concept of reciprocity. An *N*-port circuit is called *reciprocal* if its port voltages and currents satisfy the relation *∑ ik.vk*’ *= ∑ ik*’*vk,.* Here, the variables *ik* and *vk* are the port currents and voltages and currents under one set of excitations, while *ik*’ and *vk*’ exist under a different one, f*or the same internal branches of the N-port.* The summations are for all *N* ports. By Tellegen’s theorem, the same relation must then hold for the internal branches of the *N-*port [1.3].

The condition and its applications will next be analyzed for two-ports.

Consider the circuits shown in Fig. 1.17. Assume that the two-port contains only linear resistors, and no controlled sources. Regarding the circuit of Fig. 1.17(a) as ***N*** and that of Fig. 1.17(b) as ***N’***, Tellegen’s theorem gives

$$\begin{array}{c}-v\_{2}i\_{2}^{'}+\sum\_{}^{}v\_{k}i\_{k}^{'}=-v\_{2}i\_{2}^{'}+\sum\_{}^{}i\_{k}R\_{k}i\_{k}^{'}=0 \#\left(1.24\right)\end{array}$$

Here, the summations are for all *interior* branches inside the two-port. Interchanging the roles of the two circuits results in the relation

$$\begin{array}{c}-v\_{1}^{'}i\_{1}+\sum\_{}^{}v\_{k}^{'}i\_{k}=-v\_{1}^{'}i\_{1}+\sum\_{}^{}i\_{k}R\_{k}i\_{k}^{'}=0 \#\left(1.25\right)\end{array}$$

Since the summations of the terms for the interior branches are equal, subtracting the two equations gives $v\_{2}i\_{2}^{'}=v\_{1}^{'}i\_{1}$. Hence the resistive two-port is reciprocal. Also, we obtain

$$\begin{array}{c}\frac{v\_{2}}{i\_{1}}=\frac{v\_{1}^{'}}{i\_{1}^{'}} \#\left(1.26\right)\end{array}$$

The two transfer functions (*transconductances)* are thus equal. Note that in this analysis **N** and **N’** contain the same internal branches, but under different terminations. Note also that assuming reciprocity for the two-port, the relation $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k}$gives directly $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$,



 Figure 1.17: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

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Figure 1.20. Reciprocity relations for passive linear circuits. Redraw figure! With no writing

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Figure 1.21: (a) Physical network **N;** (b) Auxiliary network **N’.**

**Redo without green**

By contrast, using reciprocity the contribution of *V*1 to the output *vout* can be found from Fig. 1.21(b) using the relation

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will no longer hold for general excitations. However, a modified form of reciprocity, called *interreciprocity* may be introduced for such non-reciprocal circuits. It also offers useful applications, similar to the ones available for reciprocal two-ports. It involves the introduction of a modified version **N’** of the physical network **N,** called the *adjoint network.* It is constructed such that the reciprocity condition on the port currents and voltages

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is restored for the two networks. By Tellegen’s theorem, this requires that the condition

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also remain valid for the internal branches of **N** and **N’** under all excitations**.** Thus, **N’** must be changed so as to achieve this.

Let the two-port contain resistors and a single voltage-controlled current source located in branches *a* and *b* (Fig. 1.22(a)). Its adjoint replica, unknown at this point, is shown in Fig. 1.22(b). In order to satisfy eq. (1.31), the following relation between the voltages and currents in branches *a* and *b* in **N** and **N’** must hold

$$\begin{array}{c}i\_{a}v\_{a}^{'}+i\_{b}v\_{b}^{'}=i\_{a}^{'}v\_{a}+i\_{b}^{'}v\_{b} \#\left(1.32\right)\end{array}$$

This can be used to find the two branches in **N’** corresponding to the VCVS**.** For the VCCS, the branch relations are $i\_{a}=0$, and $i\_{b}=g\_{m}v\_{a}$*.* Substituting into (1.32) gives the reciprocity condition

 $\begin{array}{c}i\_{a}^{'}-g\_{m}v\_{b}^{'}=-i\_{b}^{'}\frac{v\_{b}}{v\_{a}} \#\left(1.33\right)\end{array}$



1. (b)

Fig. 1.22. VCCS and its adjoint circuit.

Eq. (1.33) must hold for *all* values of ${v\_{b}}/{v\_{a}}$. This is only possible if $i\_{b}^{'}=0$, and $i\_{a}^{'}=g\_{m}v\_{b}^{'}$ *ia*. The image in **N’** of the VCCS of circuit **N** is thus another VCCS, turned around. If there are several VCCS’s in **N,** then the adjoint network **N’** must contain such appropriately modified replicas (i.e., turned around VCCS stages) for all of them.

Generalizing this result, consider the branch admittance matrix ***G***connecting the internal branch voltages and XX Please fix this paragraph.

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The derivation given above can be simplified. For a two-port containing only resistors and VCCSs, eq. (1.31) may be written in the form

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{'T}Gv\_{B} \#\left(1.34\right)\end{array}$$

Since the quantities on both sides are scalars, it is permissible to take the transpose of the right side. This gives

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{T}G^{T}v\_{B}^{'}\#\left(1.35\right)\end{array}$$

and hence $G^{'}=G^{T}$, as obtained before.

A derivation similar to the one resulting in the adjoint branches of the VCCS can be performed to give the adjoint branches of all controlled sources. A table of the results is shown in Fig. 1.23. (Sh. 21 or T-LaPatra Table 9.1).



Fig. 1.23: Controlled sources with their adjoint networks.

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By the definition of **N’,** the physical network **N**and its adjoint network **N’** satisfy the reciprocity condition. Hence, the efficient analysis derived in Sec. 1.2 for multi-source active networks can be used with a slight modification, as described below:

1. Draw the adjoint circuit **N’**, setting the values of all previous independent sources to zero. Thus, voltage sources become short circuits, current sources open circuits.
2. Replace the original output signal with an independent source. If it is voltage, replace it with a current source *I*; if it is a voltage source, with a voltage source *V.* For convenience, *I* = -1 amp, of *V*= 1 volt may be used.
3. Analyze the *adjoint network* **N’**. The desired result for the output in **N** will be the weighted sum of the independent source values in **N** with the weight of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica.

Fig. 1.24 illustrates the process for a network **N** with many independent sources and an output voltage.





Figure 1.24. The application of interreciprocity for the analysis and Thevenin modeling of multi-source circuits.

Interreciprocity may also be used to obtain efficiently the Thevenin equivalent of a linear circuit with several independent as well as dependent sources. The open-circuit output *vo* is the voltage in the Thevenin model, derived as shown above.From Fig. 1.24, the output impedance *Z* of the physical network N can be obtained by setting all independent sources to 0, and finding the output voltage when the output port is excited by a -1 amp current source. By interreciprocity, $v\_{o}^{'}i\_{o}=i\_{o}^{'}v\_{o}$. Hence, the impedance in the Thevenin model is numerically equal to the output voltage of **N’,** thus $Z=Z^{'}=-v\_{o}^{'}$.

There are several more applications of interreciprocity and the adjoint network. One is noise analysis in linear or nonlinear active circuits, another is sensitivity analysis. These will be briefly discussed next.

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*1.7. Noise Analysis Using Interreciprocity*

In a CMOS integrated circuit, every transistor is affected by thermal noise caused by the random motion of the charge carriers in its channel, and by flicker noise (1/f noise) due to the trapping and releasing of these carriers. These noise effects can be modeled by a single independent noise voltage source at the gate of the device (Fig. 11a). Since there may be dozens or even hundreds of devices in circuit, the direct calculation of the overall output noise would be extremely time consuming. Interreciprocity offers an economical alternative [1.4]. For noise calculation, each transistor can be modeled with a transconductance *gm* and the drain-to-source conductance *gds* (Fig. 11b)*.* In the adjoint network, the transconductance is turned around (Fig. 11c). Using interreciprocity, the contribution of the noise voltage *vn* to the output noise power will be $j\_{n}^{'2}=\left[g\_{m}v\_{ds}^{'}\right]^{2}v\_{n}^{2}$. Here, *vn2* is the power (mean square value) of the noise. Hence, a single analysis of the adjoint network to determine the voltages across the *gds* resistors enables the designer to find the total output noise power.



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 Fig. 1.25. (a) Noisy transistor; (b) Its small-signal model; (c) Its adjoint model.

*1.8. Sensitivity Analysis Using Interreciprocity*

In the actual implementation of a circuit, the values of its components will deviate from their theoretical ones. It is important to know how the behavior of the circuit is affected by these errors. Thus, it is useful to be able to calculate the sensitivities of the key performance parameters to element value variations. The use of the adjoint network to do this efficiently was suggested by Director and Rohrer [1.5],[1.6]. Consider again a two-port containing resistors and VCCSs, and assume that all component values have a small error, causing **G** to change to **G + ∆G.** Using the adjoint network **N’** with its transconductance matrix **GT**, eq. (1.35) is replaced by

$$\begin{array}{c}v\_{B}^{T}\left[G+∆G\right]^{T}v\_{B}^{'}-v\_{B}^{T}G^{T}v\_{B}^{'}=v\_{B}^{T}∆G^{T}v\_{B}^{'}\#\left(1.36\right)\end{array}$$

Eq. (19) gives the expression $\sum\_{}^{}i\_{k}v\_{k}^{'}-\sum\_{}^{}i\_{k}^{'}v\_{k}$ for the internal branches of two-port **N.***,* By Tellegen’s theorem, this is the negative of the same expression for the port variables. In the latter, $i\_{k}^{'}$and $v\_{k}^{'}$ remain unchanged, but*ik*and *vk*change, due to the variations in the element value.Equating the two expressions, the changes in the port variables are easily found. These give the sensitivities of the output voltage or current to all circuit parameters.

The process is illustrated with an example [1.7, pp. 389-390].

Assuming a voltage output *vout* for the physical circuit **N** containing resistors and VCCSs, the process is as follows:

1. Construct the adjoint network N’, in which the values of all independent sources of N are set to zero. Resistive branches remain unchanged, controlled sources replaced by their adjoint models. At the output port, place a current source *I =* 1 V.
2. Calculate the branch voltages $v\_{k}^{'}$across all resistors. According to the discussions above, the contribution of the incremental change *∆Gk* of an admittance *Gk* to *vout* will be $v\_{k}^{'}v\_{k}∆G\_{k}$. Thus, the output sensitivity to variations in *Gk* will be ${∆v\_{out}}/{∆G\_{k}}=v\_{k}^{'}v\_{k}$.
3. To find the sensitivity to changes in the transconductance *Glm* of a VCCS between branches *m* and *l* (Fig. X) (Fig. 9.9 in T-laP), find the controlling voltages $v\_{m}^{ }$ and $v\_{l}^{'}$in N and N’. The sensitivity is ${∆v\_{out}}/{∆G\_{lm}}=v\_{l}^{'}v\_{m}$.

Note that the formula given above has a physical interpretation. The current *ik* is the result of signal transfer from the independentsources in N to branch *k,* while $i\_{k}^{'}$ the transfer in N’ from the 1 amp current source at the former output port N’. In N, this corresponds to signal transfer from branch *k* to the output. The sensitivity is the product of these two transfer functions.

Note also that all sensitivities are obtained simultaneously in one circuit analysis, that of N’.

Sensitivity calculations for other types of controlled sources can be performed in similar ways. Consider the case of a voltage-controlled current source in **N** (Fig. 20a). In the adjoint network **N’**, it is translated into a similar VCCS, but with its input and output terminals interchanged (Fig.20b) A small change in the transconductance *Gm* can be modelled by an added current source $v\_{k}∆G\_{m}$ in parallel with the main one. Using interreciprocity, the effect of this added source will be an added output term of value $v\_{l}^{'}v\_{k}∆G\_{m}$. Here, $v\_{l}^{'}$ is the controlling voltage of the VCCS in **N’**. Hence, the sensitivity to errors in *Gm* is the product of the controlling voltages in **N** and **N’.**

*References to Chapter 1*

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*1.5 Reciprocity and Its Application*

A mere four years after the publication of Tellegen’s theorem, J. L. Bordewijk (a student of Professor Tellegen’s at the Technical University of Delft) used his theorem to define and extend the concept of reciprocity. An *N*-port circuit is called *reciprocal* if its port voltages and currents satisfy the relation *∑ ik.vk*’ *= ∑ ik*’*vk,.* Here, the variables *ik* and *vk* are the port currents and voltages and currents under one set of excitations, while *ik*’ and *vk*’ exist under a different one, f*or the same internal branches of the N-port.* The summations are for all *N* ports. By Tellegen’s theorem, the same relation must then hold for the internal branches of the *N-*port [1.3].

The condition and its applications will next be analyzed for two-ports.

Consider the circuits shown in Fig. 1.17. Assume that the two-port contains only linear resistors, and no controlled sources. Regarding the circuit of Fig. 1.17(a) as ***N*** and that of Fig. 1.17(b) as ***N’***, Tellegen’s theorem gives

$$\begin{array}{c}-v\_{2}i\_{2}^{'}+\sum\_{}^{}v\_{k}i\_{k}^{'}=-v\_{2}i\_{2}^{'}+\sum\_{}^{}i\_{k}R\_{k}i\_{k}^{'}=0 \#\left(1.24\right)\end{array}$$

Here, the summations are for all *interior* branches inside the two-port. Interchanging the roles of the two circuits results in the relation

$$\begin{array}{c}-v\_{1}^{'}i\_{1}+\sum\_{}^{}v\_{k}^{'}i\_{k}=-v\_{1}^{'}i\_{1}+\sum\_{}^{}i\_{k}R\_{k}i\_{k}^{'}=0 \#\left(1.25\right)\end{array}$$

Since the summations of the terms for the interior branches are equal, subtracting the two equations gives $v\_{2}i\_{2}^{'}=v\_{1}^{'}i\_{1}$. Hence the resistive two-port is reciprocal. Also, we obtain

$$\begin{array}{c}\frac{v\_{2}}{i\_{1}}=\frac{v\_{1}^{'}}{i\_{1}^{'}} \#\left(1.26\right)\end{array}$$

The two transfer functions (*transconductances)* are thus equal. Note that in this analysis **N** and **N’** contain the same internal branches, but under different terminations. Note also that assuming reciprocity for the two-port, the relation $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k}$gives directly $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$,



 Figure 1.17: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

Consider next the circuits shown in Fig. 1.18. Assuming that the two-port is reciprocal, combining $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k} $with $i\_{2}=i\_{1}^{'}=0$ gives $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$ and ${v\_{2}}/{i\_{1}}={v\_{1}^{'}}/{i\_{2}^{'}}$. However, to use these relations for finding the relation between the transfer functions, the input impedance of **N** andthe output impedance of **N’** must also be found. Note in particular that ${v\_{2}}/{v\_{1}} $is *not* equal to ${v\_{2}^{'}}/{v\_{1}^{'}}$. In general, the forward transfer of **N e**quals the reverse transfer function of **N’**only if the source impedances at the ports are the same. This is valid for the circuits shownin Fig. 5. Analysis gives ${v\_{2}}/{v\_{1}}={i\_{1}^{'}}/{v\_{1}^{'}}$.



 Figure 1.18: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

The forward transfer of **N e**quals the reverse transfer function of **N’**only if the source impedances at the ports are the same. This is valid for the circuits shown in Figs. 1.17 and 51.19, but not for Fig. 1.18. 

Figure 1.19: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

The circuits which meet the equal-impedancetermination condition are shown in Fig. 1.20.

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Figure 1.20. Reciprocity relations for passive linear circuits. Redraw figure! With no writing

For the steady-state sine wave analysis of a linear circuit, all voltages and currents can be represented by their complex phasors, *V(jω)* and *I(jω),* and the passive elements by their impedances *Zk(jω).* Since the KCL and KVL holds for the phasors, all derivations given above for the time-domain quantities can be replicated, to give the same relations between the voltage and current phasors of reciprocal networks as the ones derived above in the time domain.

A useful application of reciprocity is in the analysis of passive networks with multiple excitations. Such circuits occur, e.g. in the design of digital-to-analog data converters (DACs). Consider the circuit of Fig. 1.21(a). It contains five independent sources. Hence, a straightforward approach to its analysis would be to use superposition. This would require five analyses, each with one of the sources active and all others set to zero. Alternatively, node analysis may be used. This would require solving simultaneous equations involving all current sources.



Figure 1.21: (a) Physical network **N;** (b) Auxiliary network **N’.**

**Redo without green**

By contrast, using reciprocity the contribution of *V*1 to the output *vout* can be found from Fig. 1.21(b) using the relation

$$\begin{array}{c}\frac{v\_{out}}{V\_{1}}=\frac{i\_{1}}{I} \#\left(1.27\right)\end{array}$$

and those of the current sources *I*k from the relations

$$\begin{array}{c}\frac{v\_{out}}{I\_{k}}=\frac{v\_{k}}{I} k=1, 2, 3, 4, 5\#\left(1.28\right)\end{array}$$

Note that this calculation requires only *one* circuit analysis, that of the circuit of Fig. 1.21(b).

Generalizing the method, the analysis requires the following steps:

1. Redraw the circuit, setting the values of all independent sources to zero. Thus, voltage sources become short circuits, current sources become open circuits.
2. Replace the output signal with an independent source. If it is voltage, replace it with a current source *I* ; if it is a current, with a voltage source *V.* For convenience, *I* = -1 amp, or *V*= 1 volt may be used.
3. Analyze the transformed network. The desired result will be the weighted sum of the independent source values in the physical circuit, with the weight factor of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica. These weight factors also give the gains of the circuit associated with each source.

*1.6. Interreciprocity and Its Application*

If the circuit contains a controlled source, the reciprocity conditions fail. Consider a linear circuit containing resistors and also the voltage-controlled current source (VCCS) shown in Fig. 1.22. Now the condition for the internal branches

$$\begin{array}{c}\sum\_{}^{}v\_{k}i\_{k}^{'}=\sum\_{}^{}v\_{k}^{'}i\_{k} \#\left(1.29\right)\end{array}$$

will no longer hold for general excitations. However, a modified form of reciprocity, called *interreciprocity* may be introduced for such non-reciprocal circuits. It also offers useful applications, similar to the ones available for reciprocal two-ports. It involves the introduction of a modified version **N’** of the physical network **N,** called the *adjoint network.* It is constructed such that the reciprocity condition on the port currents and voltages

$$\begin{array}{c}i\_{1}v\_{1}^{'}+i\_{2}v\_{2}^{'}=i\_{1}^{'}v\_{1}+i\_{2}^{'}v\_{2} \#\left(1.30\right)\end{array}$$

is restored for the two networks. By Tellegen’s theorem, this requires that the condition

$$\begin{array}{c}\sum\_{}^{}v\_{k}i\_{k}^{'}=\sum\_{}^{}v\_{k}^{'}i\_{k} \#\left(1.31\right)\end{array}$$

also remain valid for the internal branches of **N** and **N’** under all excitations**.** Thus, **N’** must be changed so as to achieve this.

Let the two-port contain resistors and a single voltage-controlled current source located in branches *a* and *b* (Fig. 1.22(a)). Its adjoint replica, unknown at this point, is shown in Fig. 1.22(b). In order to satisfy eq. (1.31), the following relation between the voltages and currents in branches *a* and *b* in **N** and **N’** must hold

$$\begin{array}{c}i\_{a}v\_{a}^{'}+i\_{b}v\_{b}^{'}=i\_{a}^{'}v\_{a}+i\_{b}^{'}v\_{b} \#\left(1.32\right)\end{array}$$

This can be used to find the two branches in **N’** corresponding to the VCVS**.** For the VCCS, the branch relations are $i\_{a}=0$, and $i\_{b}=g\_{m}v\_{a}$*.* Substituting into (1.32) gives the reciprocity condition

 $\begin{array}{c}i\_{a}^{'}-g\_{m}v\_{b}^{'}=-i\_{b}^{'}\frac{v\_{b}}{v\_{a}} \#\left(1.33\right)\end{array}$



1. (b)

Fig. 1.22. VCCS and its adjoint circuit.

Eq. (1.33) must hold for *all* values of ${v\_{b}}/{v\_{a}}$. This is only possible if $i\_{b}^{'}=0$, and $i\_{a}^{'}=g\_{m}v\_{b}^{'}$ *ia*. The image in **N’** of the VCCS of circuit **N** is thus another VCCS, turned around. If there are several VCCS’s in **N,** then the adjoint network **N’** must contain such appropriately modified replicas (i.e., turned around VCCS stages) for all of them.

Generalizing this result, consider the branch admittance matrix ***G***connecting the internal branch voltages and XX Please fix this paragraph.

currents in **N** according to the relation $i\_{B}=Gv\_{B}$. For a two-port containing only resistors and VCCS stages,the resistive branches will be represented in ***G*** by the diagonal elements 1/*R*1 , 1/*R2* , …. A VCCS *gmi* in which branch *m* controls the current in branch *l* introduces a 0 at the location (*m,l*) and the entry *gmi* in (*l,m*). Its replica VCCS in **N’** introduces a 0 at the location (*l,m*) and the entry *gmi* in (*m,l*). This occurs for all VCCS blocks in **N.** Clearly, the branch admittance matrix ***G’*** of ***N’*** is the transpose of ***G.***

The derivation given above can be simplified. For a two-port containing only resistors and VCCSs, eq. (1.31) may be written in the form

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{'T}Gv\_{B} \#\left(1.34\right)\end{array}$$

Since the quantities on both sides are scalars, it is permissible to take the transpose of the right side. This gives

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and hence $G^{'}=G^{T}$, as obtained before.

A derivation similar to the one resulting in the adjoint branches of the VCCS can be performed to give the adjoint branches of all controlled sources. A table of the results is shown in Fig. 1.23. (Sh. 21 or T-LaPatra Table 9.1).



Fig. 1.23: Controlled sources with their adjoint networks.

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By the definition of **N’,** the physical network **N**and its adjoint network **N’** satisfy the reciprocity condition. Hence, the efficient analysis derived in Sec. 1.2 for multi-source active networks can be used with a slight modification, as described below:

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Fig. 1.24 illustrates the process for a network **N** with many independent sources and an output voltage.





Figure 1.24. The application of interreciprocity for the analysis and Thevenin modeling of multi-source circuits.

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There are several more applications of interreciprocity and the adjoint network. One is noise analysis in linear or nonlinear active circuits, another is sensitivity analysis. These will be briefly discussed next.

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 Fig. 1.25. (a) Noisy transistor; (b) Its small-signal model; (c) Its adjoint model.

*1.8. Sensitivity Analysis Using Interreciprocity*

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A mere four years after the publication of Tellegen’s theorem, J. L. Bordewijk (a student of Professor Tellegen’s at the Technical University of Delft) used his theorem to define and extend the concept of reciprocity. An *N*-port circuit is called *reciprocal* if its port voltages and currents satisfy the relation *∑ ik.vk*’ *= ∑ ik*’*vk,.* Here, the variables *ik* and *vk* are the port currents and voltages and currents under one set of excitations, while *ik*’ and *vk*’ exist under a different one, f*or the same internal branches of the N-port.* The summations are for all *N* ports. By Tellegen’s theorem, the same relation must then hold for the internal branches of the *N-*port [1.3].

The condition and its applications will next be analyzed for two-ports.

Consider the circuits shown in Fig. 1.17. Assume that the two-port contains only linear resistors, and no controlled sources. Regarding the circuit of Fig. 1.17(a) as ***N*** and that of Fig. 1.17(b) as ***N’***, Tellegen’s theorem gives

$$\begin{array}{c}-v\_{2}i\_{2}^{'}+\sum\_{}^{}v\_{k}i\_{k}^{'}=-v\_{2}i\_{2}^{'}+\sum\_{}^{}i\_{k}R\_{k}i\_{k}^{'}=0 \#\left(1.24\right)\end{array}$$

Here, the summations are for all *interior* branches inside the two-port. Interchanging the roles of the two circuits results in the relation

$$\begin{array}{c}-v\_{1}^{'}i\_{1}+\sum\_{}^{}v\_{k}^{'}i\_{k}=-v\_{1}^{'}i\_{1}+\sum\_{}^{}i\_{k}R\_{k}i\_{k}^{'}=0 \#\left(1.25\right)\end{array}$$

Since the summations of the terms for the interior branches are equal, subtracting the two equations gives $v\_{2}i\_{2}^{'}=v\_{1}^{'}i\_{1}$. Hence the resistive two-port is reciprocal. Also, we obtain

$$\begin{array}{c}\frac{v\_{2}}{i\_{1}}=\frac{v\_{1}^{'}}{i\_{1}^{'}} \#\left(1.26\right)\end{array}$$

The two transfer functions (*transconductances)* are thus equal. Note that in this analysis **N** and **N’** contain the same internal branches, but under different terminations. Note also that assuming reciprocity for the two-port, the relation $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k}$gives directly $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$,



 Figure 1.17: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

Consider next the circuits shown in Fig. 1.18. Assuming that the two-port is reciprocal, combining $\sum\_{}^{}i\_{k}v\_{k}^{'}=\sum\_{}^{}i\_{k}^{'}v\_{k} $with $i\_{2}=i\_{1}^{'}=0$ gives $i\_{1}v\_{1}^{'}= i\_{2}^{'}v\_{2}$ and ${v\_{2}}/{i\_{1}}={v\_{1}^{'}}/{i\_{2}^{'}}$. However, to use these relations for finding the relation between the transfer functions, the input impedance of **N** andthe output impedance of **N’** must also be found. Note in particular that ${v\_{2}}/{v\_{1}} $is *not* equal to ${v\_{2}^{'}}/{v\_{1}^{'}}$. In general, the forward transfer of **N e**quals the reverse transfer function of **N’**only if the source impedances at the ports are the same. This is valid for the circuits shownin Fig. 5. Analysis gives ${v\_{2}}/{v\_{1}}={i\_{1}^{'}}/{v\_{1}^{'}}$.



 Figure 1.18: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

The forward transfer of **N e**quals the reverse transfer function of **N’**only if the source impedances at the ports are the same. This is valid for the circuits shown in Figs. 1.17 and 51.19, but not for Fig. 1.18. 

Figure 1.19: (a) Terminations for N; (b) Terminations for N’. Redraw without green lines.

The circuits which meet the equal-impedancetermination condition are shown in Fig. 1.20.

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Figure 1.20. Reciprocity relations for passive linear circuits. Redraw figure! With no writing

For the steady-state sine wave analysis of a linear circuit, all voltages and currents can be represented by their complex phasors, *V(jω)* and *I(jω),* and the passive elements by their impedances *Zk(jω).* Since the KCL and KVL holds for the phasors, all derivations given above for the time-domain quantities can be replicated, to give the same relations between the voltage and current phasors of reciprocal networks as the ones derived above in the time domain.

A useful application of reciprocity is in the analysis of passive networks with multiple excitations. Such circuits occur, e.g. in the design of digital-to-analog data converters (DACs). Consider the circuit of Fig. 1.21(a). It contains five independent sources. Hence, a straightforward approach to its analysis would be to use superposition. This would require five analyses, each with one of the sources active and all others set to zero. Alternatively, node analysis may be used. This would require solving simultaneous equations involving all current sources.



Figure 1.21: (a) Physical network **N;** (b) Auxiliary network **N’.**

**Redo without green**

By contrast, using reciprocity the contribution of *V*1 to the output *vout* can be found from Fig. 1.21(b) using the relation

$$\begin{array}{c}\frac{v\_{out}}{V\_{1}}=\frac{i\_{1}}{I} \#\left(1.27\right)\end{array}$$

and those of the current sources *I*k from the relations

$$\begin{array}{c}\frac{v\_{out}}{I\_{k}}=\frac{v\_{k}}{I} k=1, 2, 3, 4, 5\#\left(1.28\right)\end{array}$$

Note that this calculation requires only *one* circuit analysis, that of the circuit of Fig. 1.21(b).

Generalizing the method, the analysis requires the following steps:

1. Redraw the circuit, setting the values of all independent sources to zero. Thus, voltage sources become short circuits, current sources become open circuits.
2. Replace the output signal with an independent source. If it is voltage, replace it with a current source *I* ; if it is a current, with a voltage source *V.* For convenience, *I* = -1 amp, or *V*= 1 volt may be used.
3. Analyze the transformed network. The desired result will be the weighted sum of the independent source values in the physical circuit, with the weight factor of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica. These weight factors also give the gains of the circuit associated with each source.

*1.6. Interreciprocity and Its Application*

If the circuit contains a controlled source, the reciprocity conditions fail. Consider a linear circuit containing resistors and also the voltage-controlled current source (VCCS) shown in Fig. 1.22. Now the condition for the internal branches

$$\begin{array}{c}\sum\_{}^{}v\_{k}i\_{k}^{'}=\sum\_{}^{}v\_{k}^{'}i\_{k} \#\left(1.29\right)\end{array}$$

will no longer hold for general excitations. However, a modified form of reciprocity, called *interreciprocity* may be introduced for such non-reciprocal circuits. It also offers useful applications, similar to the ones available for reciprocal two-ports. It involves the introduction of a modified version **N’** of the physical network **N,** called the *adjoint network.* It is constructed such that the reciprocity condition on the port currents and voltages

$$\begin{array}{c}i\_{1}v\_{1}^{'}+i\_{2}v\_{2}^{'}=i\_{1}^{'}v\_{1}+i\_{2}^{'}v\_{2} \#\left(1.30\right)\end{array}$$

is restored for the two networks. By Tellegen’s theorem, this requires that the condition

$$\begin{array}{c}\sum\_{}^{}v\_{k}i\_{k}^{'}=\sum\_{}^{}v\_{k}^{'}i\_{k} \#\left(1.31\right)\end{array}$$

also remain valid for the internal branches of **N** and **N’** under all excitations**.** Thus, **N’** must be changed so as to achieve this.

Let the two-port contain resistors and a single voltage-controlled current source located in branches *a* and *b* (Fig. 1.22(a)). Its adjoint replica, unknown at this point, is shown in Fig. 1.22(b). In order to satisfy eq. (1.31), the following relation between the voltages and currents in branches *a* and *b* in **N** and **N’** must hold

$$\begin{array}{c}i\_{a}v\_{a}^{'}+i\_{b}v\_{b}^{'}=i\_{a}^{'}v\_{a}+i\_{b}^{'}v\_{b} \#\left(1.32\right)\end{array}$$

This can be used to find the two branches in **N’** corresponding to the VCVS**.** For the VCCS, the branch relations are $i\_{a}=0$, and $i\_{b}=g\_{m}v\_{a}$*.* Substituting into (1.32) gives the reciprocity condition

 $\begin{array}{c}i\_{a}^{'}-g\_{m}v\_{b}^{'}=-i\_{b}^{'}\frac{v\_{b}}{v\_{a}} \#\left(1.33\right)\end{array}$



1. (b)

Fig. 1.22. VCCS and its adjoint circuit.

Eq. (1.33) must hold for *all* values of ${v\_{b}}/{v\_{a}}$. This is only possible if $i\_{b}^{'}=0$, and $i\_{a}^{'}=g\_{m}v\_{b}^{'}$ *ia*. The image in **N’** of the VCCS of circuit **N** is thus another VCCS, turned around. If there are several VCCS’s in **N,** then the adjoint network **N’** must contain such appropriately modified replicas (i.e., turned around VCCS stages) for all of them.

Generalizing this result, consider the branch admittance matrix ***G***connecting the internal branch voltages and XX Please fix this paragraph.

currents in **N** according to the relation $i\_{B}=Gv\_{B}$. For a two-port containing only resistors and VCCS stages,the resistive branches will be represented in ***G*** by the diagonal elements 1/*R*1 , 1/*R2* , …. A VCCS *gmi* in which branch *m* controls the current in branch *l* introduces a 0 at the location (*m,l*) and the entry *gmi* in (*l,m*). Its replica VCCS in **N’** introduces a 0 at the location (*l,m*) and the entry *gmi* in (*m,l*). This occurs for all VCCS blocks in **N.** Clearly, the branch admittance matrix ***G’*** of ***N’*** is the transpose of ***G.***

The derivation given above can be simplified. For a two-port containing only resistors and VCCSs, eq. (1.31) may be written in the form

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{'T}Gv\_{B} \#\left(1.34\right)\end{array}$$

Since the quantities on both sides are scalars, it is permissible to take the transpose of the right side. This gives

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{T}G^{T}v\_{B}^{'}\#\left(1.35\right)\end{array}$$

and hence $G^{'}=G^{T}$, as obtained before.

A derivation similar to the one resulting in the adjoint branches of the VCCS can be performed to give the adjoint branches of all controlled sources. A table of the results is shown in Fig. 1.23. (Sh. 21 or T-LaPatra Table 9.1).



Fig. 1.23: Controlled sources with their adjoint networks.

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By the definition of **N’,** the physical network **N**and its adjoint network **N’** satisfy the reciprocity condition. Hence, the efficient analysis derived in Sec. 1.2 for multi-source active networks can be used with a slight modification, as described below:

1. Draw the adjoint circuit **N’**, setting the values of all previous independent sources to zero. Thus, voltage sources become short circuits, current sources open circuits.
2. Replace the original output signal with an independent source. If it is voltage, replace it with a current source *I*; if it is a voltage source, with a voltage source *V.* For convenience, *I* = -1 amp, of *V*= 1 volt may be used.
3. Analyze the *adjoint network* **N’**. The desired result for the output in **N** will be the weighted sum of the independent source values in **N** with the weight of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica.

Fig. 1.24 illustrates the process for a network **N** with many independent sources and an output voltage.





Figure 1.24. The application of interreciprocity for the analysis and Thevenin modeling of multi-source circuits.

Interreciprocity may also be used to obtain efficiently the Thevenin equivalent of a linear circuit with several independent as well as dependent sources. The open-circuit output *vo* is the voltage in the Thevenin model, derived as shown above.From Fig. 1.24, the output impedance *Z* of the physical network N can be obtained by setting all independent sources to 0, and finding the output voltage when the output port is excited by a -1 amp current source. By interreciprocity, $v\_{o}^{'}i\_{o}=i\_{o}^{'}v\_{o}$. Hence, the impedance in the Thevenin model is numerically equal to the output voltage of **N’,** thus $Z=Z^{'}=-v\_{o}^{'}$.

There are several more applications of interreciprocity and the adjoint network. One is noise analysis in linear or nonlinear active circuits, another is sensitivity analysis. These will be briefly discussed next.

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*1.7. Noise Analysis Using Interreciprocity*

In a CMOS integrated circuit, every transistor is affected by thermal noise caused by the random motion of the charge carriers in its channel, and by flicker noise (1/f noise) due to the trapping and releasing of these carriers. These noise effects can be modeled by a single independent noise voltage source at the gate of the device (Fig. 11a). Since there may be dozens or even hundreds of devices in circuit, the direct calculation of the overall output noise would be extremely time consuming. Interreciprocity offers an economical alternative [1.4]. For noise calculation, each transistor can be modeled with a transconductance *gm* and the drain-to-source conductance *gds* (Fig. 11b)*.* In the adjoint network, the transconductance is turned around (Fig. 11c). Using interreciprocity, the contribution of the noise voltage *vn* to the output noise power will be $j\_{n}^{'2}=\left[g\_{m}v\_{ds}^{'}\right]^{2}v\_{n}^{2}$. Here, *vn2* is the power (mean square value) of the noise. Hence, a single analysis of the adjoint network to determine the voltages across the *gds* resistors enables the designer to find the total output noise power.



REDRAW (capital N on top)

 Fig. 1.25. (a) Noisy transistor; (b) Its small-signal model; (c) Its adjoint model.

*1.8. Sensitivity Analysis Using Interreciprocity*

In the actual implementation of a circuit, the values of its components will deviate from their theoretical ones. It is important to know how the behavior of the circuit is affected by these errors. Thus, it is useful to be able to calculate the sensitivities of the key performance parameters to element value variations. The use of the adjoint network to do this efficiently was suggested by Director and Rohrer [1.5],[1.6]. Consider again a two-port containing resistors and VCCSs, and assume that all component values have a small error, causing **G** to change to **G + ∆G.** Using the adjoint network **N’** with its transconductance matrix **GT**, eq. (1.35) is replaced by

$$\begin{array}{c}v\_{B}^{T}\left[G+∆G\right]^{T}v\_{B}^{'}-v\_{B}^{T}G^{T}v\_{B}^{'}=v\_{B}^{T}∆G^{T}v\_{B}^{'}\#\left(1.36\right)\end{array}$$

Eq. (19) gives the expression $\sum\_{}^{}i\_{k}v\_{k}^{'}-\sum\_{}^{}i\_{k}^{'}v\_{k}$ for the internal branches of two-port **N.***,* By Tellegen’s theorem, this is the negative of the same expression for the port variables. In the latter, $i\_{k}^{'}$and $v\_{k}^{'}$ remain unchanged, but*ik*and *vk*change, due to the variations in the element value.Equating the two expressions, the changes in the port variables are easily found. These give the sensitivities of the output voltage or current to all circuit parameters.

The process is illustrated with an example [1.7, pp. 389-390].

Assuming a voltage output *vout* for the physical circuit **N** containing resistors and VCCSs, the process is as follows:

1. Construct the adjoint network N’, in which the values of all independent sources of N are set to zero. Resistive branches remain unchanged, controlled sources replaced by their adjoint models. At the output port, place a current source *I =* 1 V.
2. Calculate the branch voltages $v\_{k}^{'}$across all resistors. According to the discussions above, the contribution of the incremental change *∆Gk* of an admittance *Gk* to *vout* will be $v\_{k}^{'}v\_{k}∆G\_{k}$. Thus, the output sensitivity to variations in *Gk* will be ${∆v\_{out}}/{∆G\_{k}}=v\_{k}^{'}v\_{k}$.
3. To find the sensitivity to changes in the transconductance *Glm* of a VCCS between branches *m* and *l* (Fig. X) (Fig. 9.9 in T-laP), find the controlling voltages $v\_{m}^{ }$ and $v\_{l}^{'}$in N and N’. The sensitivity is ${∆v\_{out}}/{∆G\_{lm}}=v\_{l}^{'}v\_{m}$.

Note that the formula given above has a physical interpretation. The current *ik* is the result of signal transfer from the independentsources in N to branch *k,* while $i\_{k}^{'}$ the transfer in N’ from the 1 amp current source at the former output port N’. In N, this corresponds to signal transfer from branch *k* to the output. The sensitivity is the product of these two transfer functions.

Note also that all sensitivities are obtained simultaneously in one circuit analysis, that of N’.

Sensitivity calculations for other types of controlled sources can be performed in similar ways. Consider the case of a voltage-controlled current source in **N** (Fig. 20a). In the adjoint network **N’**, it is translated into a similar VCCS, but with its input and output terminals interchanged (Fig.20b) A small change in the transconductance *Gm* can be modelled by an added current source $v\_{k}∆G\_{m}$ in parallel with the main one. Using interreciprocity, the effect of this added source will be an added output term of value $v\_{l}^{'}v\_{k}∆G\_{m}$. Here, $v\_{l}^{'}$ is the controlling voltage of the VCCS in **N’**. Hence, the sensitivity to errors in *Gm* is the product of the controlling voltages in **N** and **N’.**

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*1.5 Reciprocity and Its Application*

A mere four years after the publication of Tellegen’s theorem, J. L. Bordewijk (a student of Professor Tellegen’s at the Technical University of Delft) used his theorem to define and extend the concept of reciprocity. An *N*-port circuit is called *reciprocal* if its port voltages and currents satisfy the relation *∑ ik.vk*’ *= ∑ ik*’*vk,.* Here, the variables *ik* and *vk* are the port currents and voltages and currents under one set of excitations, while *ik*’ and *vk*’ exist under a different one, f*or the same internal branches of the N-port.* The summations are for all *N* ports. By Tellegen’s theorem, the same relation must then hold for the internal branches of the *N-*port [1.3].

The condition and its applications will next be analyzed for two-ports.

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Here, the summations are for all *interior* branches inside the two-port. Interchanging the roles of the two circuits results in the relation

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Since the summations of the terms for the interior branches are equal, subtracting the two equations gives $v\_{2}i\_{2}^{'}=v\_{1}^{'}i\_{1}$. Hence the resistive two-port is reciprocal. Also, we obtain

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**Redo without green**

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is restored for the two networks. By Tellegen’s theorem, this requires that the condition

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also remain valid for the internal branches of **N** and **N’** under all excitations**.** Thus, **N’** must be changed so as to achieve this.

Let the two-port contain resistors and a single voltage-controlled current source located in branches *a* and *b* (Fig. 1.22(a)). Its adjoint replica, unknown at this point, is shown in Fig. 1.22(b). In order to satisfy eq. (1.31), the following relation between the voltages and currents in branches *a* and *b* in **N** and **N’** must hold

$$\begin{array}{c}i\_{a}v\_{a}^{'}+i\_{b}v\_{b}^{'}=i\_{a}^{'}v\_{a}+i\_{b}^{'}v\_{b} \#\left(1.32\right)\end{array}$$

This can be used to find the two branches in **N’** corresponding to the VCVS**.** For the VCCS, the branch relations are $i\_{a}=0$, and $i\_{b}=g\_{m}v\_{a}$*.* Substituting into (1.32) gives the reciprocity condition

 $\begin{array}{c}i\_{a}^{'}-g\_{m}v\_{b}^{'}=-i\_{b}^{'}\frac{v\_{b}}{v\_{a}} \#\left(1.33\right)\end{array}$



1. (b)

Fig. 1.22. VCCS and its adjoint circuit.

Eq. (1.33) must hold for *all* values of ${v\_{b}}/{v\_{a}}$. This is only possible if $i\_{b}^{'}=0$, and $i\_{a}^{'}=g\_{m}v\_{b}^{'}$ *ia*. The image in **N’** of the VCCS of circuit **N** is thus another VCCS, turned around. If there are several VCCS’s in **N,** then the adjoint network **N’** must contain such appropriately modified replicas (i.e., turned around VCCS stages) for all of them.

Generalizing this result, consider the branch admittance matrix ***G***connecting the internal branch voltages and XX Please fix this paragraph.

currents in **N** according to the relation $i\_{B}=Gv\_{B}$. For a two-port containing only resistors and VCCS stages,the resistive branches will be represented in ***G*** by the diagonal elements 1/*R*1 , 1/*R2* , …. A VCCS *gmi* in which branch *m* controls the current in branch *l* introduces a 0 at the location (*m,l*) and the entry *gmi* in (*l,m*). Its replica VCCS in **N’** introduces a 0 at the location (*l,m*) and the entry *gmi* in (*m,l*). This occurs for all VCCS blocks in **N.** Clearly, the branch admittance matrix ***G’*** of ***N’*** is the transpose of ***G.***

The derivation given above can be simplified. For a two-port containing only resistors and VCCSs, eq. (1.31) may be written in the form

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{'T}Gv\_{B} \#\left(1.34\right)\end{array}$$

Since the quantities on both sides are scalars, it is permissible to take the transpose of the right side. This gives

$$\begin{array}{c}v\_{B}^{T}G^{'}v\_{B}^{'}=v\_{B}^{T}G^{T}v\_{B}^{'}\#\left(1.35\right)\end{array}$$

and hence $G^{'}=G^{T}$, as obtained before.

A derivation similar to the one resulting in the adjoint branches of the VCCS can be performed to give the adjoint branches of all controlled sources. A table of the results is shown in Fig. 1.23. (Sh. 21 or T-LaPatra Table 9.1).



Fig. 1.23: Controlled sources with their adjoint networks.

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By the definition of **N’,** the physical network **N**and its adjoint network **N’** satisfy the reciprocity condition. Hence, the efficient analysis derived in Sec. 1.2 for multi-source active networks can be used with a slight modification, as described below:

1. Draw the adjoint circuit **N’**, setting the values of all previous independent sources to zero. Thus, voltage sources become short circuits, current sources open circuits.
2. Replace the original output signal with an independent source. If it is voltage, replace it with a current source *I*; if it is a voltage source, with a voltage source *V.* For convenience, *I* = -1 amp, of *V*= 1 volt may be used.
3. Analyze the *adjoint network* **N’**. The desired result for the output in **N** will be the weighted sum of the independent source values in **N** with the weight of each current source the voltage across its zero valued replica, and that of each voltage source the current flowing across its short-circuit replica.

Fig. 1.24 illustrates the process for a network **N** with many independent sources and an output voltage.





Figure 1.24. The application of interreciprocity for the analysis and Thevenin modeling of multi-source circuits.

Interreciprocity may also be used to obtain efficiently the Thevenin equivalent of a linear circuit with several independent as well as dependent sources. The open-circuit output *vo* is the voltage in the Thevenin model, derived as shown above.From Fig. 1.24, the output impedance *Z* of the physical network N can be obtained by setting all independent sources to 0, and finding the output voltage when the output port is excited by a -1 amp current source. By interreciprocity, $v\_{o}^{'}i\_{o}=i\_{o}^{'}v\_{o}$. Hence, the impedance in the Thevenin model is numerically equal to the output voltage of **N’,** thus $Z=Z^{'}=-v\_{o}^{'}$.

There are several more applications of interreciprocity and the adjoint network. One is noise analysis in linear or nonlinear active circuits, another is sensitivity analysis. These will be briefly discussed next.

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*1.7. Noise Analysis Using Interreciprocity*

In a CMOS integrated circuit, every transistor is affected by thermal noise caused by the random motion of the charge carriers in its channel, and by flicker noise (1/f noise) due to the trapping and releasing of these carriers. These noise effects can be modeled by a single independent noise voltage source at the gate of the device (Fig. 11a). Since there may be dozens or even hundreds of devices in circuit, the direct calculation of the overall output noise would be extremely time consuming. Interreciprocity offers an economical alternative [1.4]. For noise calculation, each transistor can be modeled with a transconductance *gm* and the drain-to-source conductance *gds* (Fig. 11b)*.* In the adjoint network, the transconductance is turned around (Fig. 11c). Using interreciprocity, the contribution of the noise voltage *vn* to the output noise power will be $j\_{n}^{'2}=\left[g\_{m}v\_{ds}^{'}\right]^{2}v\_{n}^{2}$. Here, *vn2* is the power (mean square value) of the noise. Hence, a single analysis of the adjoint network to determine the voltages across the *gds* resistors enables the designer to find the total output noise power.



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 Fig. 1.25. (a) Noisy transistor; (b) Its small-signal model; (c) Its adjoint model.

*1.8. Sensitivity Analysis Using Interreciprocity*

In the actual implementation of a circuit, the values of its components will deviate from their theoretical ones. It is important to know how the behavior of the circuit is affected by these errors. Thus, it is useful to be able to calculate the sensitivities of the key performance parameters to element value variations. The use of the adjoint network to do this efficiently was suggested by Director and Rohrer [1.5],[1.6]. Consider again a two-port containing resistors and VCCSs, and assume that all component values have a small error, causing **G** to change to **G + ∆G.** Using the adjoint network **N’** with its transconductance matrix **GT**, eq. (1.35) is replaced by

$$\begin{array}{c}v\_{B}^{T}\left[G+∆G\right]^{T}v\_{B}^{'}-v\_{B}^{T}G^{T}v\_{B}^{'}=v\_{B}^{T}∆G^{T}v\_{B}^{'}\#\left(1.36\right)\end{array}$$

Eq. (19) gives the expression $\sum\_{}^{}i\_{k}v\_{k}^{'}-\sum\_{}^{}i\_{k}^{'}v\_{k}$ for the internal branches of two-port **N.***,* By Tellegen’s theorem, this is the negative of the same expression for the port variables. In the latter, $i\_{k}^{'}$and $v\_{k}^{'}$ remain unchanged, but*ik*and *vk*change, due to the variations in the element value.Equating the two expressions, the changes in the port variables are easily found. These give the sensitivities of the output voltage or current to all circuit parameters.

The process is illustrated with an example [1.7, pp. 389-390].

Assuming a voltage output *vout* for the physical circuit **N** containing resistors and VCCSs, the process is as follows:

1. Construct the adjoint network N’, in which the values of all independent sources of N are set to zero. Resistive branches remain unchanged, controlled sources replaced by their adjoint models. At the output port, place a current source *I =* 1 V.
2. Calculate the branch voltages $v\_{k}^{'}$across all resistors. According to the discussions above, the contribution of the incremental change *∆Gk* of an admittance *Gk* to *vout* will be $v\_{k}^{'}v\_{k}∆G\_{k}$. Thus, the output sensitivity to variations in *Gk* will be ${∆v\_{out}}/{∆G\_{k}}=v\_{k}^{'}v\_{k}$.
3. To find the sensitivity to changes in the transconductance *Glm* of a VCCS between branches *m* and *l* (Fig. X) (Fig. 9.9 in T-laP), find the controlling voltages $v\_{m}^{ }$ and $v\_{l}^{'}$in N and N’. The sensitivity is ${∆v\_{out}}/{∆G\_{lm}}=v\_{l}^{'}v\_{m}$.

Note that the formula given above has a physical interpretation. The current *ik* is the result of signal transfer from the independentsources in N to branch *k,* while $i\_{k}^{'}$ the transfer in N’ from the 1 amp current source at the former output port N’. In N, this corresponds to signal transfer from branch *k* to the output. The sensitivity is the product of these two transfer functions.

Note also that all sensitivities are obtained simultaneously in one circuit analysis, that of N’.

Sensitivity calculations for other types of controlled sources can be performed in similar ways. Consider the case of a voltage-controlled current source in **N** (Fig. 20a). In the adjoint network **N’**, it is translated into a similar VCCS, but with its input and output terminals interchanged (Fig.20b) A small change in the transconductance *Gm* can be modelled by an added current source $v\_{k}∆G\_{m}$ in parallel with the main one. Using interreciprocity, the effect of this added source will be an added output term of value $v\_{l}^{'}v\_{k}∆G\_{m}$. Here, $v\_{l}^{'}$ is the controlling voltage of the VCCS in **N’**. Hence, the sensitivity to errors in *Gm* is the product of the controlling voltages in **N** and **N’.**

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